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The General Uncertainty Relation for Real Signals in Communication Theory*

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The general uncertainty relation for real time functions in communication theory is derived. The product of pulse duration and spectral width referred to the positive frequency spectrum only, is not less than 1.1802..., as compared with 2 in the Heisenberg and Gabor cases. This minimum is reached with a pulse whose time and spectral functions are numerically evaluated.

1. INTRODUCTION

It has often been pointed out that Heisenberg's uncertainty relation has an important analogue in communication theory. Here the complementary quantities are the duration of a pulse and the width of its spectrum. Both are defined with the aid of the variances of the squares of the absolute values of the time and spectral functions, respectively, and these functions are connected by a Fourier transform just as are the corresponding ones in quantum theory.

If, as has been done in many texts, the mathematics used in quantum theory are transferred unchanged to the problem in communication theory, discrepancies arise between theory and practice for the following reasons: In the calculation of the spectral width, the variance is referred to the centroid of the square of the spectral function, and the whole spectrum, extending from $-\infty$ to $+\infty$, is taken into account. In communication theory one

* This paper is a section of a more comprehensive study concerning optimal pulses and uncertainty relations given in two internal reports of the Research Institute, Ulm, spring 1969. In addition to the material presented, the minimum products and optimum pulse shapes under various conditions (ratio of bandwidth to middle frequency, etc.) are further discussed there. They can be found starting with the differential equation (18), which subsequently is less specialized than is done here in (27). To give a paper of a reasonable length, these results, seeming to have a somewhat lesser importance, were omitted.

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has to deal only with real time functions. So the spectrum for negative frequencies is uniquely determined by that for positive frequencies and has the same absolute value. Thus the centroid is always zero. But obviously it suffices and, moreover, corresponds better to measuring practice, to use only positive frequencies in the definition of the spectral width. This has been realized by several authors, e.g., Gabor, 1946, Kay and Silverman, 1957, and Rothe, 1962. Gabor was the first one to find, on this basis, a way to obtain reasonable results for pulses modulated on a carrier. His theory, however, necessitates the introduction of complex time functions, which is again unsatisfactory. Using the real time function and positive frequencies, Kay and Silverman, 1957, obtained an inequality, from which Gabor's results also follow. Unexpected difficulties arise when an attempt is made to find the greatest lower bound of the product of pulse and spectral widths from the inequality. Examples show that pulses exist with a product less than that which corresponds to the greatest lower bound in Heisenberg's and Gabor's theory. Since by definition the product cannot be negative, it is clear that a greatest lower bound exists. It remains an open question, however, whether it is zero or greater than zero, or whether it is a minimum, and if so, which function gives the minimum. Of course, as stated by Kay and Silverman, a greatest lower bound of value zero would go against one's physical intuition. Indeed, it would mean the nonexistence of an uncertainty relation in communication theory.

We shall show in the following sections that the open questions can be answered completely in a different way, based on the principles of the calculus of variations and the theory of differential equations. Because of the situation outlined above, the existence questions had to be dealt with somewhat more extensively than is usually necessary with technical problems. In the technical field, the existence of a solution is often warranted for physical reasons, but here we would have to rely on physical intuition only.

In the following sections it is shown that a positive minimum of the pulse duration–bandwidth product exists, its value is calculated, and the corresponding time and spectral functions are determined. They are no Gaussian curves.

2. Definitions and the Uncertainty Relation for Odd Pulses

Let the relation between a time function $f(t)$ and its spectrum $F(\omega)$ be given by the Fourier pair

$$f(t) = \int_{-\infty}^{+\infty} F(\omega)\, e^{i\omega t}\, d\omega,$$
$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t}\, dt. \tag{1}$$

Parseval's formula then takes the form

$$\int_{-\infty}^{+\infty} |F(\omega)|^2\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |f(t)|^2\, dt = \frac{1}{2\pi} E. \tag{2}$$

In the following let $f(t)$ be real. Then

$$F(-\omega) = F^*(\omega), \tag{3}$$

and

$$E = 4\pi \int_0^{\infty} |F(\omega)|^2\, d\omega. \tag{4}$$

$$\bar{\omega} = \frac{\int_0^{\infty} \omega\, |F(\omega)|^2\, d\omega}{\int_0^{\infty} |F(\omega)|^2\, d\omega} \quad \text{and} \quad \bar{t} = \frac{\int_{-\infty}^{+\infty} t\, |f(t)|^2\, dt}{\int_{-\infty}^{+\infty} |f(t)|^2\, dt}$$

are the usual definitions of the first moments of $|F(\omega)|^2$ and $|f(t)|^2$. By a suitable choice of the origin of the time axis $\bar{t}$ can always be made zero.

The second moment of $|f(t)|^2$ referred to the centroid $\bar{t} = 0$,

$$\overline{(t - \bar{t})^2} = \overline{t^2} = \frac{1}{E} \int_{-\infty}^{+\infty} t^2\, |f(t)|^2\, dt,$$

may be expressed in terms of $F(\omega)$,

$$\overline{(t - \bar{t})^2} = \frac{4\pi}{E} \int_0^{\infty} \left| \frac{dF(\omega)}{d\omega} \right|^2 d\omega. \tag{5}$$

The second moment of $|F(\omega)|^2$ referred to an arbitrary ω_0 is given by

$$\overline{(\omega - \omega_0)^2} = \frac{4\pi}{E} \int_0^{\infty} (\omega - \omega_0)^2\, |F(\omega)|^2\, d\omega. \tag{6}$$

Introducing the pulse widths Δt and $\Delta\omega$, respectively, by

$$\Delta t = 2s_t = 2\sqrt{\overline{(t - \bar{t})^2}}, \qquad \Delta\omega = 2s_\omega = 2\sqrt{\overline{(\omega - \omega_0)^2}},$$

the result obtained by Kay and Silverman, 1957, may be written

$$\Delta t \cdot \Delta\omega \geqslant 2\left[1 - \frac{\omega_0 \,|\, F(0)|^2}{\int_0^\infty |\, F(\omega)|^2 \, d\omega}\right]. \tag{7}$$

As discussed by the authors, for $\omega_0 = \bar{\omega}$ the equality sign does not hold.

For odd pulses $F(0) = 0$, and for $\omega_0 = \bar{\omega}$ (7) gives

$$\Delta t \cdot \Delta\omega > 2. \tag{8}$$

The lower bound 2 can be approached as closely as desired by the function

$$F(\omega) = iC\omega e^{-(\omega-\omega_0)^2/2a^2}, \ \omega \geqslant 0; \qquad F(\omega) = F^*(-\omega), \ \omega \leqslant 0,$$

C and a being real constants. Thus (8) is the uncertainty relation for odd pulses.

As has already been stated by Kay and Silverman, no greatest lower bound can be derived from (7) for more general pulses. In the following, we shall first obtain a greatest lower bound for even pulses and then combine the results for even and odd pulses to find the general uncertainty relation.

3. A New Approach

3.1. *Formulation of the Problem*

We shall seek the minimum of the product

$$\overline{(t - \bar{t})^2} \; \overline{(\omega - \omega_0)^2} = \text{min.}$$

With the definitions (5), (6), and (4) this becomes

$$P[F] \equiv \frac{\int_0^\infty |\, dF/d\omega \,|^2 \, d\omega \int_0^\infty (\omega - \omega_0)^2 \,|\, F \,|^2 \, d\omega}{(\int_0^\infty |\, F \,|^2 \, d\omega)^2} = \text{min.} \tag{9}$$

Differentiation with respect to the arbitrary parameter ω_0 shows that with a given F a minimum of $P[F]$ is reached if

$$\int_0^\infty (\omega - \omega_0) \,|\, F \,|^2 \, d\omega = 0; \tag{10}$$

i.e., if ω_0 is the centroid of $|\, F \,|^2$.

Furthermore, it is easily seen that

$$\omega = k\omega_1 , \qquad t = \frac{1}{k} t_1$$

transforms the Fourier pair (1) into the Fourier pair

$$g(t_1) = f\left(\frac{t_1}{k}\right), \qquad G(\omega_1) = k \cdot F(k\omega_1).$$

The new centroid $\bar{\omega}_1$ is then

$$\bar{\omega}_1 = \frac{\bar{\omega}}{k} .$$

The variance of $|g(t_1)|^2$ is k^2 times that of $|f(t)|^2$, the variance of $|G(\omega_1)|^2$ is $1/k^2$ times that of $|F(\omega)|^2$, and the product of the variances remains unchanged. So for every $F(\omega)$ with $\bar{\omega} > 0$ there exists a $G(\omega)$ with the same product of the variances and a centroid at a point $\bar{\omega}_1$, whose position can be chosen at will. Thus when solving (9) with a fixed ω_0 instead of $\bar{\omega}$, the solution F automatically has the property that ω_0 is the centroid of $|F|^2$.

First we shall consider even and odd time functions $f(t)$ only. Then $F(\omega)$ is real or purely imaginary. As only $|F|^2$ and $|F'|^2$ appear in (9), there is no longer a difference between the two cases and we may assume F to be real. The general case will be treated in the last section.

The function $F(\omega)$ will be assumed to be of class C^2, i.e., to have continuous second derivatives in the interval $(0, \infty)$ and to vanish for $\omega \to \infty$ in such a manner that the three integrals in (9) exist.

3.2. *The Differential Equation*

For any function $F(\omega)$ not identically zero the three integrals in (9) are positive and so is the value of $P[F]$. Therefore, instead of $P[F]$, its logarithm may be minimized, the logarithm being a monotonically increasing function of positive arguments. Then (9) becomes

$$\ln \int_0^\infty \left| \frac{dF}{d\omega} \right|^2 d\omega + \ln \int_0^\infty (\omega - \omega_0)^2 |F|^2 \, d\omega - 2 \ln \int_0^\infty |F|^2 \, d\omega = \min. \tag{11}$$

Although neither (9) nor (11) is a variational problem in the proper sense, we can proceed according to the principles used in the calculus of variations: Assuming that a solution $F_0(\omega)$ of the problem exists (it should be kept in mind that the results of the following considerations in this chapter do not automatically ensure the validity of this assumption; therefore it is of greatest importance to prove this point subsequently in Section 4), we form the function

$$F_0(\omega) + \epsilon \cdot \eta(\omega), \tag{12}$$

depending linearly on the parameter ϵ. Here $\eta(\omega)$ is an arbitrary function of class C^2 with the boundary values $\eta(0) = 0$ and $\eta(\infty) = 0$, and behaves for $\omega \to \infty$ in such a way that again the integrals in (11) exist, when formed with (12). If (12) is inserted in (11) instead of F, the derivative with respect to ϵ must vanish for $\epsilon = 0$. This leads to

$$\frac{\int_0^\infty F_0'\eta'\,d\omega}{\int_0^\infty F_0'^2\,d\omega} + \frac{\int_0^\infty (\omega-\omega_0)^2 F_0\eta\,d\omega}{\int_0^\infty (\omega-\omega_0)^2 F_0^2\,d\omega} - 2\frac{\int_0^\infty F_0\eta\,d\omega}{\int_0^\infty F_0^2\,d\omega} = 0. \tag{13}$$

Partial integration gives

$$\int_0^\infty F_0'\eta'\,d\omega = F_0'\eta\Big|_0^\infty - \int_0^\infty F_0''\eta\,d\omega = -\int_0^\infty F_0''\eta\,d\omega. \tag{14}$$

As $P[F]$ does not change, if F is multiplied by a constant, we may assume

$$\int_0^\infty F_0^2\,d\omega = 1. \tag{15}$$

Furthermore, we introduce abbreviations for the other two integrals not containing η,

$$C_1^2 = \int_0^\infty F_0'^2\,d\omega \qquad\qquad > 0, \tag{16}$$

$$C_2^2 = \int_0^\infty (\omega-\omega_0)^2 F_0^2\,d\omega > 0. \tag{17}$$

For convenience, we choose $C_1 > 0$, $C_2 > 0$ also. With (14), (15), (16), and (17) we obtain from (13) the differential equation

$$F_0''(\omega) - \frac{C_1^2}{C_2^2}(\omega-\omega_0)^2 F_0(\omega) + 2C_1^2 F_0(\omega) = 0 \tag{18}$$

in the usual manner.

3.3. *Boundary Conditions*

The solutions of the differential equation (18) must be valid for $0 \leqslant \omega \leqslant \infty$. At the lower boundary nothing is known at this stage about $F_0(0)$ and $F_0'(0)$. At the upper boundary $\omega \to \infty$ we can make use of the asymptotic expansion of the solutions of (18), which can be found by the usual method,

$$F_0(\omega) \sim \omega^\rho e^{\pm \frac{C_1(\omega-\omega_0)^2}{2C_2}} \cdot \sum_{v=0}^{\infty} b_v \omega^{-v}, \tag{19}$$

where

$$\rho = \mp C_1 C_2 - \tfrac{1}{2}, \tag{20}$$

b_0 is an arbitrary constant, and the following b_v may be obtained by recursion. In (19) and (20) the upper and lower signs belong together.

Because of (15), it follows from (19) that $F_0(\infty) = 0$ and also that $F_0'(\infty) = 0$.

3.4. *Further Necessary Conditions*

Multiplication of (18) by $F_0(\omega)$ and integration over positive ω yields

$$\int_0^\infty F_0'' F_0 \, d\omega - \frac{C_1^2}{C_2^2} \int_0^\infty (\omega - \omega_0)^2 F_0^2 \, d\omega + 2C_1^2 \int_0^\infty F_0^2 \, d\omega = 0. \tag{21}$$

The first term may be integrated by parts,

$$\int_0^\infty F_0'' F_0 \, d\omega = F_0' F_0 \Big|_0^\infty - \int_0^\infty F_0'^2 \, d\omega. \tag{22}$$

From the boundary condition $F_0(\infty) = 0$ it follows that the first term on the right side of (22) vanishes at the upper boundary and with (15), (16), and (17) we obtain from (21)

$$F_0'(0) \cdot F_0(0) = 0 \tag{23}$$

as a necessary condition.

A further necessary condition can be derived in a similar way. Multiplication of (18) by $F_0'(\omega)$ and integration over positive ω gives

$$\int_0^\infty F_0'' F_0' \, d\omega - \frac{C_1^2}{C_2^2} \int_0^\infty (\omega - \omega_0)^2 F_0 F_0' \, d\omega + 2C_1^2 \int_0^\infty F_0 F_0' \, d\omega = 0.$$

Here all terms can be integrated by parts,

$$\frac{1}{2}F_0'^2\Big|_0^\infty - \frac{C_1^2}{C_2^2}\left[(\omega-\omega_0)^2\frac{F_0^2}{2}\Big|_0^\infty - \int_0^\infty(\omega-\omega_0)F_0^2\,d\omega\right] + 2C_1^2\frac{F_0^2}{2}\Big|_0^\infty = 0.$$

Because of (10) the integral is zero and we have

$$\frac{1}{2}F_0'^2(0) + \frac{C_1^2}{C_2^2}\omega_0^2\frac{F_0^2(0)}{2} - C_1^2F_0^2(0) = 0. \tag{24}$$

If $F_0(0)$ were zero, then according to (24) $F_0'(0)$ would also be zero, and from (18) it would follow by continued differentiation that all derivatives of F_0, and thus F_0 itself, would be zero. These findings, of course, are in agreement with (8), since the greatest lower bound cannot be reached. So a minimum and a corresponding function F do not exist. Thus only the solution with $F_0(0) \neq 0$ remains. (23) and (24) now yield

$$C_2^2 = \frac{\omega_0^2}{2}. \tag{25}$$

Insertion in (18) gives

$$F_0''(\omega) - 2\frac{C_1^2}{\omega_0^2}(\omega^2 - 2\omega_0\omega)F_0(\omega) = 0.$$

As discussed in Section 3.1, ω_0 can be chosen at will, so we take $\omega_0 = 1$ and obtain

$$F_0''(\omega) - 2C_1^2\omega(\omega-2)F_0(\omega) = 0. \tag{26}$$

From (25) we further have

$$C_2^2 = \tfrac{1}{2}$$

and

$$\Delta t \cdot \Delta\omega = 4C_1C_2 = 2\sqrt{2}\,C_1.$$

The differential equation (26) has a solution fulfilling the boundary conditions $F_0'(0) = 0$, $F_0(\infty) = 0$ only if $2C_1^2 = \frac{1}{4}(\Delta t \cdot \Delta\omega)^2$ is an eigenvalue λ of the boundary value problem

$$F'' - \lambda\omega(\omega-2)F = 0, \qquad F'(0) = 0, \qquad F(\infty) = 0. \tag{27}$$

It can be shown that this problem has a countably infinite number of positive eigenvalues (see the Appendix). Thus, if a solution of the minimum problem exists, the least positive eigenvalue λ_0 of (27) yields the smallest product,

$$(\Delta t \cdot \Delta\omega)_{\min} = 2\sqrt{\lambda_0} = 4C_1C_2 = 2\sqrt{2}\,C_1\,. \tag{28}$$

The eigenfunction F_0 belonging to λ_0 represents the spectrum of the pulse by which the minimum product is reached. F_0 decreases monotonically from a finite value for $\omega = 0$ and approaches the ω axis asymptotically for $\omega \to \infty$ (Appendix).

It should be mentioned that by a suitable transformation (27) can be transformed into a standard form of Weber's equation. Thus $F_0(\omega)$ is a parabolic cylinder function. However, it was not found useful to go back to this theory here.

4. The Existence of a Solution of the Problem

Now the possibility is given to evaluate immediately the eigenfunctions and eigenvalues of (27). But we would not be sure that the results obtained are the solution of our problem, for the assumption is still unproved that the minimum problem (9) actually has a solution. This is not obvious from physical reasons (compare the discussion by Kay and Silverman, 1957). If a minimizing function should not exist, we could have found subordinate solutions. Then it should be possible to give examples having a smaller product, i.e., our calculations above would be rather useless. In addition to the necessary existence proof we have to show further, that the solution of the eigenvalue problem, which is uniquely determined by (27) and (15), obeys the relations (10), (16), and (17), where (10) and (17) with $\omega_0 = 1$ now read

$$\int_0^\infty (\omega - 1) F_0^2\, d\omega = 0 \tag{29}$$

and

$$\int_0^\infty (\omega - 1)^2 F_0^2\, d\omega = \tfrac{1}{2}. \tag{30}$$

Both questions may be dispatched in the following manner: We set

$$u = \frac{\int_0^\infty F'^2\, d\omega}{\int_0^\infty F^2\, d\omega}\,, \tag{31}$$

$$v = \frac{\int_0^\infty (\omega - \omega_0)^2 F^2\, d\omega}{\int_0^\infty F^2\, d\omega}\,, \tag{32}$$

and make u a minimum under the boundary conditions

$$F'(0) = 0, \qquad F(\infty) = 0, \tag{33}$$

and under the additional condition $v = \text{const.}$

This leads to the variational problem

$$\frac{\int_0^\infty F'^2\,d\omega + \mu' \int_0^\infty (\omega - \omega_0)^2 F^2\,d\omega}{\int_0^\infty F^2\,d\omega} = \min \tag{34}$$

with the Euler differential equation

$$F'' - \mu'(\omega - \omega_0)^2 F + \lambda' F = 0 \tag{35}$$

and the boundary conditions (33).

Having solved the variational problem, the parameter μ' must be determined so that the additional condition is met.

If the minimum problem (9) has a solution, it must coincide with one of the solutions of (34), viz., with that for a certain v, and thus also for a certain μ'. According to the remarks made in Section 3.1, it must be contained among those solutions for which the centroid of F^2 is ω_0.

Introducing

$$\frac{\omega}{\omega_0} = x \tag{36}$$

and setting

$$F(\omega) = y(x), \tag{37}$$

$$\mu'\omega_0^4 = \mu, \qquad \lambda'\omega_0^2 = \lambda, \tag{38}$$

$$\frac{\int_0^\infty y'^2\,dx}{\int_0^\infty y^2\,dx} = u_1, \qquad \frac{\int_0^\infty (x-1)^2 y^2\,dx}{\int_0^\infty y^2\,dx} = v_1 \tag{39}$$

(34), (35), and (33) become

$$u_1 + \mu v_1 = \min, \tag{40}$$

$$y'' - \mu(x-1)^2 y + \lambda y = 0, \tag{41}$$

$$y'(0) = 0, \qquad y(\infty) = 0. \tag{42}$$

Admitting all functions $y(x)$ of class C^2 meeting the boundary conditions

(42), and assuming the parameter μ to be positive, the eigenvalue problem (41), (42) is self-adjoint and definite, i.e., with

$$L[y] \equiv -y'' + \mu(x-1)^2 y$$

we have

$$\int_0^\infty yL[y]\,dx > 0 \tag{43}$$

for all admissible $y \neq 0$, as may be seen by integration by parts and consideration of the boundary conditions. Furthermore, $\lambda = 0$ is not an eigenvalue, for if an eigenfunction y_v belonging to the eigenvalue λ_v is inserted in (43),

$$0 < \int_0^\infty y_v L[y_v]\,dx = \lambda_v \int_0^\infty y_v{}^2\,dx$$

results because of (41) and $\lambda_v \neq 0$ follows.

On these premises the variational problem (40) possesses a solution, Kamke, 1939 and 1961, and the value of the minimum is the least positive eigenvalue $\lambda_0(\mu)$ of (41). Its existence is proved in the Appendix for a neighborhood of $\mu = \lambda_0$, which is sufficient for our purposes. The function yielding the minimum in (40) is the eigenfunction $y_0(x, \mu)$ belonging to $\lambda_0(\mu)$. Thus for all admissible y,

$$\frac{\int_0^\infty y'^2\,dx + \mu \int_0^\infty (x-1)^2 y^2\,dx}{\int_0^\infty y^2\,dx} \geqslant \lambda_0(\mu),$$

or with the abbreviations (39),

$$u_1 + \mu v_1 \geqslant \lambda_0(\mu). \tag{44}$$

The proof given by Kamke and Collatz, 1939, originally intended only for finite boundary values, may be applied verbally. It is based on the "alternative," i.e., the theorem that the inhomogeneous boundary value problem is always solvable if the corresponding homogeneous boundary problem has no solution except the trivial one.

The validity of the proof commonly given for the alternative, Bieberbach, 1956, is likewise restricted to finite boundary values, but can easily be extended to the present case if use is made of the asymptotic expansion for y (of (18), (19), (20) with suitably changed parameters).

We shall now derive a further relation between u_1 and v_1. We multiply the differential equation (41) by y' and integrate from 0 to x.

$$\int_0^x y'y''\,dx - \mu \int_0^x (\xi - 1)^2 yy'\,d\xi + \lambda \int_0^x yy'\,d\xi = 0. \tag{45}$$

It follows that

$$\tfrac{1}{2}y'^2 - \tfrac{1}{2}\mu(x-1)^2 y^2 + \tfrac{1}{2}\mu y^2(0) + \mu \int_0^x (\xi - 1)\,y^2\,d\xi + \tfrac{1}{2}\lambda y^2 - \tfrac{1}{2}\lambda y^2(0) = 0, \tag{46}$$

and for $x \to \infty$

$$\tfrac{1}{2}(\lambda - \mu)\,y^2(0) = \mu \int_0^\infty (\xi - 1)\,y^2\,d\xi. \tag{47}$$

Addition of (46) and (47) gives

$$\tfrac{1}{2}y'^2 - \tfrac{1}{2}\mu(x-1)^2 y^2 - \mu \int_x^\infty (\xi - 1)\,y^2\,d\xi + \tfrac{1}{2}\lambda y^2 = 0.$$

Integrating again from 0 to ∞, we obtain

$$\begin{aligned} &\tfrac{1}{2}\int_0^\infty y'^2\,dx - \tfrac{1}{2}\mu \int_0^\infty (x-1)^2 y^2\,dx \\ &\qquad - \mu \int_0^\infty \int_x^\infty (\xi - 1)\,y^2\,d\xi\,dx + \tfrac{1}{2}\lambda \int_0^\infty y^2\,dx = 0. \end{aligned} \tag{48}$$

Integration by parts of the third term yields

$$\begin{aligned} &-\mu \int_x^\infty (\xi - 1)\,y^2\,d\xi \cdot x \,\Big|_0^\infty + \mu \int_0^\infty -(x-1)\,y^2 \cdot x\,dx \\ &\quad = -\mu \int_0^\infty x(x-1)\,y^2\,dx, \end{aligned}$$

since

$$-\mu \lim_{x\to\infty} \frac{\int_x^\infty (\xi - 1)\,y^2\,d\xi}{1/x} = 0.$$

Thus (48) becomes

$$\int_0^\infty y'^2\,dx - 3\mu \int_0^\infty (x-1)^2 y^2\,dx - 2\mu \int_0^\infty (x-1)\,y^2\,dx + \lambda \int_0^\infty y^2\,dx = 0. \tag{49}$$

Referring to the above remark, we may now pick out those functions of the set of admissible functions for which the centroid of F^2 is ω_0. For y this means, according to (36), (37),

$$\int_0^\infty (x - 1)\, y^2\, dx = 0.$$

Then

$$\mu = \lambda$$

follows from (47), and (41), (42), (44), and (49) become

$$y'' - \lambda x(x - 2) y = 0, \qquad y'(0) = 0, \qquad y(\infty) = 0. \tag{50}$$

$$u_1 + \lambda_0 v_1 \geqslant \lambda_0, \tag{51}$$

$$u_1 - 3\lambda v_1 + \lambda = 0. \tag{52}$$

Here λ is an arbitrary positive eigenvalue, λ_0 is the least positive eigenvalue of (50), and the equality sign in (51) applies only for the eigenfunction y_0 belonging to λ_0.

If now in (51) the equality sign is taken, and (52) is written for λ_0, two linear equations are obtained for u_1 and v_1, whose solution is

$$u_1 = \frac{\lambda_0}{2}, \qquad v_1 = \frac{1}{2}. \tag{53}$$

Thus these values of u_1 and v_1 are assumed for the eigenfunction y_0 belonging to the smallest positive eigenvalue λ_0 of (50). y_0 makes (40) with $\lambda = \lambda_0$ a minimum and the value of the minimum likewise is λ_0. The centroid of y_0^2 is $x = 1$.

Now the missing demonstration of the validity of (29), (30), and (16) can be given at once: If in (41), (42) $\mu = \lambda$ is set from the start, (27) results; the steps (45)–(47) give (29), the calculation following (47) leads to (52). (51) may be arrived at as above or, with the equality sign, by multiplication of the differential equation by y, subsequent integration from 0 to ∞, and consideration of the boundary values. With (51) and (52) one obtains (53) again and with regard to (28) this is identical with (16) and (30).

We now transfer the results to the function $F_0(\omega)$, which corresponds to y_0. Because of (31), (32), (36), and (37) we have with (53),

$$u = \frac{1}{\omega_0^2}\, u_1 = \frac{\lambda_0}{2\omega_0^2}, \qquad v = \omega_0^2 v_1 = \frac{\omega_0^2}{2}. \tag{54}$$

According to (36), (37), (50), and (38) $F_0(\omega)$ is the eigenfunction of

$$F'' - \lambda' \cdot \frac{1}{\omega_0^2}\,\omega(\omega - 2\omega_0)F = 0, \qquad F'(0) = 0, \quad F(\infty) = 0,$$

which belongs to the least positive eigenvalue

$$\lambda_0' = \frac{\lambda_0}{\omega_0^2}\,.$$

F_0 makes

$$u + \frac{\lambda_0'}{\omega_0^2}\,v = u + \frac{\lambda_0}{\omega_0^4}\,v$$

a minimum. The value of the minimum is λ_0/ω_0^2. Thus

$$u + \frac{\lambda_0}{\omega_0^4}\,v \geqslant \frac{\lambda_0}{\omega_0^2} \tag{55}$$

for all admissible F. The centroid of F_0^2 is ω_0. So F_0 solves the problem to minimize u if $v = \omega_0^2/2$ is kept constant. The product $u \cdot v$ results from (54),

$$u \cdot v = \frac{\lambda_0}{4}, \tag{56}$$

independent of ω_0 and thus independent of v. So F_0 also solves the minimum problem (9) and (56) gives the least possible product of the variances.

The results can be illustrated by making u and v coordinates of a rectangular coordinate system (Fig. 1). If ω_0 is given, a point $u, v(\omega_0)$ in the first quadrant of the u–v plane is obtained for every admissible function F. Since v depends only on ω_0, points belonging to the same function F, but to different values of ω_0, lie on a parallel to the v axis. The lowest of these points is obtained if ω_0 is the centroid of F^2. If the equality sign is taken in (55), a set of straight lines is obtained with the parameter ω_0. In Fig. 1 the straight lines for $\omega_0 = 1$ and $\omega_0 = \frac{1}{2}$ are shown. The inequality (55) means that no admissible functions exist for which $u, v(\omega_0)$ lies to the left of and below the straight line (in the shaded area in Fig. 1). The solution of the problem $u =$ Min, $v =$ const is given by the point of intersection of two straight lines: The first one is the limiting straight line of (55). The second one is obtained from (52) with $\lambda = \lambda_0$, if u_1 and v_1 are replaced by u and v in accordance with (54). Its equation reads

$$\omega_0^2 u - 3\lambda_0 \frac{v}{\omega_0^2} + \lambda_0 = 0.$$

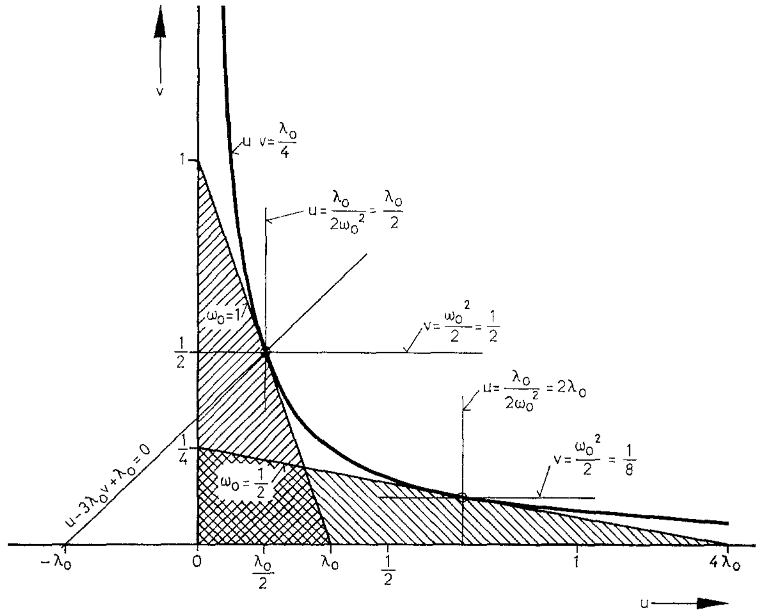

FIG. 1. u–v Plane.

In Fig. 1 it is also drawn for $\omega_0 = 1$. The point of intersection, A, halves that part of the first straight line that is bounded by the axes. A is the only point on this line that belongs to an admissible function. It is obvious that A represents the solution of the problem $u = \min$, $v = \text{const}$, since all points to the left of A on the line $v = \text{const}$ passing through A lie in the "forbidden" area. The points below A on the parallel to the v axis through A also lie in the "forbidden" area. This shows that for the function F corresponding to A, ω_0 is the centroid of F^2. All points of intersection lie on the hyperbola $u \cdot v = \lambda_0/4$, which is the envelope of the set of limiting straight lines of (55).

5. Numerical Evaluation

After the existence of a function F_0 giving a minimum product $P_{\min}$ has been established, a calculation of F_0 can now easily be carried out on a computer. The differential equation (27) was solved with the initial values $F(0) = 1, F'(0) = 0$ and an approximate value for λ_0 obtained on an analog computer. Improved values for λ_0 and values of the corresponding eigenfunctions of F were then found with the aid of a digital computer.

From $\lambda_0 = 0.348\,225...$ the minimum product

$$P_{\min} = (\Delta t \cdot \Delta\omega)_{\min} = 1.180\,212...$$

is obtained. The eigenfunction F is shown in Fig. 2a and Table 1 gives more accurate values. It should be noted that F is normalized to $F(0) = 1$, not to $\int_0^\infty F^2\,d\omega = 1$. If the latter normalization is desired, the values in Table 1 and Fig. 2a should be multiplied by $[\int_0^\infty F^2\,d\omega]^{-1} = 0.585\,514$. Figure 2b gives comparisons of $F(\omega)$ with two Gaussian curves having the same amplitude and derivative at the origin and a second common point

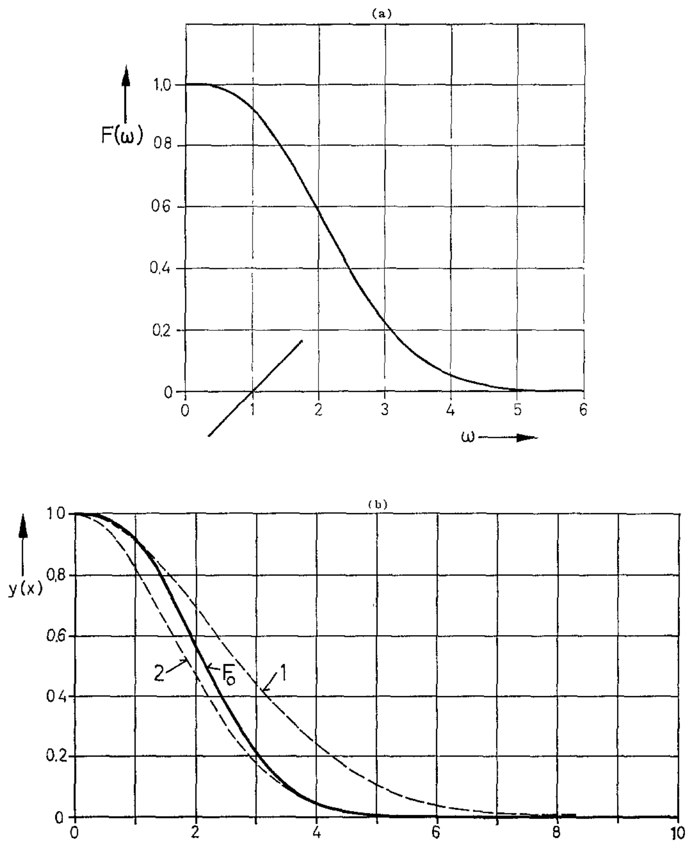

FIG. 2. (a) Optimum spectral function $F(\omega)$ as the first eigenfunction of (27); (b) Comparison of $F(\omega)$ with two Gaussian pulses.

for low and high values of ω. A considerable difference exists in spite of the first intuitive impression to see a curve $F(\omega)$ just like a Gaussian pulse. Figure 3 shows the time function $f(t)$ belonging to $F(\omega)$ according to (1) and Table 2 gives some more accurate values.

TABLE 1

x	$y(x)$
0	1.000 000 00
0.2	$9.991\ 179\ 84 \times 10^{-1}$
0.4	$9.933\ 228\ 61 \times 10^{-1}$
0.6	$9.787\ 768\ 07 \times 10^{-1}$
0.8	$9.528\ 879\ 42 \times 10^{-1}$
1.0	$9.143\ 655\ 73 \times 10^{-1}$
1.2	$8.632\ 064\ 06 \times 10^{-1}$
1.4	$8.005\ 867\ 46 \times 10^{-1}$
1.6	$7.286\ 581\ 23 \times 10^{-1}$
1.8	$6.502\ 641\ 39 \times 10^{-1}$
2.0	$5.686\ 114\ 37 \times 10^{-1}$
2.2	$4.869\ 354\ 09 \times 10^{-1}$
2.4	$4.082\ 008\ 00 \times 10^{-1}$
2.6	$3.348\ 695\ 28 \times 10^{-1}$
2.8	$2.687\ 552\ 77 \times 10^{-1}$
3.0	$2.109\ 696\ 33 \times 10^{-1}$
3.2	$1.619\ 509\ 66 \times 10^{-1}$
3.4	$1.215\ 571\ 27 \times 10^{-1}$
3.6	$8.919\ 779\ 15 \times 10^{-2}$
3.8	$6.398\ 185\ 73 \times 10^{-2}$
4.0	$4.485\ 884\ 22 \times 10^{-2}$
4.2	$3.073\ 923\ 16 \times 10^{-2}$
4.4	$2.058\ 551\ 61 \times 10^{-2}$
4.6	$1.347\ 188\ 60 \times 10^{-2}$
4.8	$8.615\ 299\ 03 \times 10^{-3}$
5.0	$5.383\ 539\ 18 \times 10^{-3}$
6.0	$3.621\ 149\ 90 \times 10^{-4}$
7.0	$1.361\ 050\ 81 \times 10^{-5}$
8.0	$6.516\ 738\ 52 \times 10^{-7}$

Computed for $\Delta t \cdot \Delta w = 1.180\ 212\ 2499$

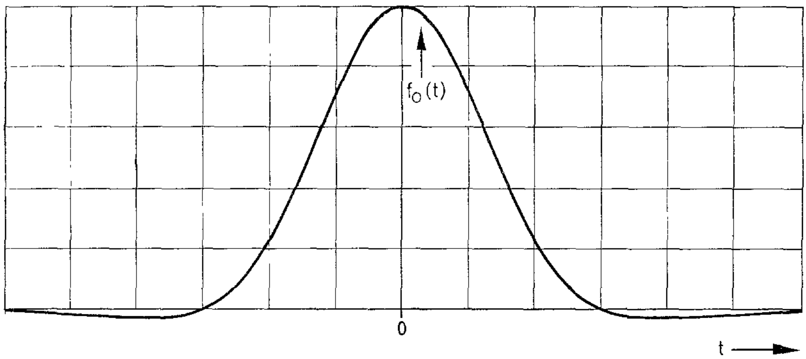

FIG. 3. The time function $f(t)$ belonging to $F(\omega)$ in Fig. 2a.

TABLE 2

t	$f(t)$
0	2.244 374 1
0.1	2.214 944 3
0.2	2.128 686 8
0.3	1.991 481 0
0.4	1.812 379 1
0.5	1.603 046 8
0.6	1.375 861 3
0.7	1.143 539 0
0.8	$9.176\,464\,1 \times 10^{-1}$
0.9	$7.078\,101\,2 \times 10^{-1}$
1.0	$5.212\,448\,4 \times 10^{-1}$
1.1	$3.620\,872\,7 \times 10^{-1}$
1.2	$2.322\,751\,0 \times 10^{-1}$
1.3	$1.311\,189\,9 \times 10^{-1}$
1.4	$5.620\,879\,9 \times 10^{-2}$
1.5	$3.946\,177\,1 \times 10^{-3}$
1.6	$-2.981\,644\,3 \times 10^{-2}$
1.7	$-4.944\,889\,2 \times 10^{-2}$
1.8	$-5.841\,070\,5 \times 10^{-2}$
1.9	$-6.038\,973\,7 \times 10^{-2}$
2.0	$-5.784\,659\,6 \times 10^{-2}$

6. The General Uncertainty Relation

So far we have considered only even and odd time functions and found that for even functions a minimum m of the product of the variances is reached, whereas for odd functions there exists only a greatest lower bound l, which is higher than the minimum for the even functions, $l > m$. It still remains to prove that also in the general case the product of the variances cannot be less than the minimum m. For this purpose we write $f(t)$ in a well-known manner as the sum of its even and odd parts,

$$f(t) = g(t) + u(t),$$

with

$$g(t) = \tfrac{1}{2}[f(t) + f(-t)],$$
$$u(t) = \tfrac{1}{2}[f(t) - f(-t)].$$

Then

$$\int_{-\infty}^{\infty} t^2 f^2(t)\, dt = \int_{-\infty}^{\infty} t^2 (g+u)^2\, dt = \int_{-\infty}^{\infty} t^2 g^2\, dt + \int_{-\infty}^{\infty} t^2 u^2\, dt \qquad (57)$$

and

$$\int_{-\infty}^{\infty} f^2(t)\, dt = \int_{-\infty}^{\infty} (g+u)^2\, dt = \int_{-\infty}^{\infty} g^2\, dt + \int_{-\infty}^{\infty} u^2\, dt. \qquad (58)$$

With (3) we have

$$f(-t) = \int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega t}\, d\omega = \int_{-\infty}^{\infty} F(-\omega)\, e^{i\omega t}\, d\omega = \int_{-\infty}^{\infty} F^*(\omega)\, e^{i\omega t}\, d\omega$$

and the spectra of g and u become

$$G(\omega) = \tfrac{1}{2}[F(\omega) + F^*(\omega)] = \operatorname{Re} F(\omega)$$

and

$$iU(\omega) = \tfrac{1}{2}[F(\omega) - F^*(\omega)] = i \operatorname{Im} F(\omega).$$

respectively.

Thus

$$\begin{aligned} \int_0^{\infty} (\omega - \omega_0)^2 \,|F|^2\, d\omega &= \int_0^{\infty} (\omega - \omega_0)^2 \,|\operatorname{Re} F(\omega) + i \operatorname{Im} F(\omega)|^2\, d\omega \\ &= \int_0^{\infty} (\omega - \omega_0)^2\, [G^2(\omega) + U^2(\omega)]\, d\omega \end{aligned} \qquad (59)$$

and

$$\int_0^\infty |F|^2\, d\omega = \int_0^\infty [G^2(\omega) + U^2(\omega)]\, d\omega. \tag{60}$$

For the product of the variances it follows with (57), (58), (59), and (60) that

$$\frac{[\int_{-\infty}^{\infty} t^2 g^2\, dt + \int_{-\infty}^{\infty} t^2 u^2\, dt] \cdot [4\pi \int_0^\infty (\omega - \omega_0)^2\, G^2\, d\omega + 4\pi \int_0^\infty (\omega - \omega_0)^2\, U^2\, d\omega]}{[\int_{-\infty}^{\infty} g^2\, dt + \int_{-\infty}^{\infty} u^2\, dt] \cdot [4\pi \int_0^\infty G^2\, d\omega + 4\pi \int_0^\infty U^2\, d\omega]}$$

$$= \frac{(p_1 + p_2)(q_1 + q_2)}{(n_1 + n_2)^2},$$

where abbreviations have been introduced for the various terms in an unmistakable manner. Further it has been taken into account that according to (3) the terms in the denominator are equal in pairs.

We already know that

$$\frac{p_1 q_1}{n_1^2} \geqslant m \qquad \text{and} \qquad \frac{p_1 q_2}{n_2^2} \geqslant l > m \tag{61}$$

everywhere, and we wish to prove that then

$$\frac{(p_1 + p_2)(q_1 + q_2)}{(n_1 + n_2)^2} \geqslant m,$$

or

$$p_1 q_1 + p_2 q_1 + p_1 q_2 + p_2 q_2 \geqslant m(n_1^2 + n_2^2 + 2 n_1 n_2).$$

According to (61), it obviously suffices to show that

$$\tfrac{1}{2}(p_2 q_1 + p_1 q_2) \geqslant m n_1 n_2,$$

or, since the geometric mean is always less than the arithmetic mean, that

$$\sqrt{p_2 q_1 p_1 q_2} \geqslant m n_1 n_2 .$$

This, however, follows again from (61). Now we can combine the various results and state the general uncertainty relation for real signals,

$$\Delta t \cdot \Delta\omega \geqslant 1.180\,212\,2\ldots .$$

APPENDIX: On the Eigenvalues and Eigenfunctions of the Boundary Problems (27) and (41), (42)

Consider the behaviour of the solutions of

$$y'' = \lambda x(x-2)y \tag{A.1}$$

with the initial values $y(0) \neq 0$, $y'(0) = 0$ for $x \geqslant 0$ and λ increasing from zero. $y(0)$ may be assumed positive. For $\lambda = 0$ the solution is the straight line $y = y(0)$ (Fig. 4). The solution varies continuously with λ. For $\lambda > 0$ y'' vanishes for $x = 0$, $x = 2$, and $y = 0$. Here points of inflection of y are to be expected.

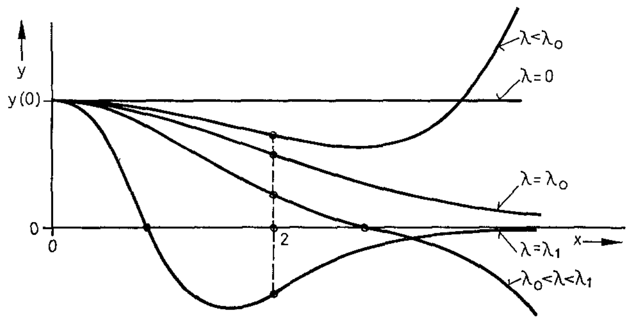

Fig. 4. The behaviour of the solutions of (A.1).

For $0 < x < 2$ the sign of y'' equals that of $-y$ and $y(x)$ is concave toward the x axis. For $x > 2$ y'' has the same sign as y and y is convex toward the x axis. In connection with the asymptotic behaviour of the solutions of (A.1) (cf. (18), (19), (20)) it follows from these remarks that the solutions behave as sketched in Fig. 4: If λ increases from zero, y is at first positive for all $x > 0$ and tends to infinity for $x \to \infty$; then a value $\lambda = \lambda_0$, the least positive eigenvalue, is reached, for which y_0, the eigenfunction belonging to it, approaches the x axis asymptotically. y_0 is likewise positive for all $x > 0$ and decreases monotonically. If λ further increases, a zero of y appears in the region $x > 2$ and $y \to -\infty$ for $x \to \infty$. This zero shifts to smaller values of x into the region $0 < x < 2$. Then a value $\lambda = \lambda_1$, the second eigenvalue, is reached, and y_1, the corresponding eigenfunction, approaches the x axis from below. Thus y_1 has one zero for $x > 0$ in the region $0 < x < 2$. If λ further increases, a second zero appears for $x > 2$,

$y \to +\infty$ for $x \to \infty$, the zero shifts into the region $0 < x < 2$, and so forth. So there exists a countable sequence of eigenvalues $\lambda_0, \lambda_1, \ldots$, and the corresponding eigenfunctions y_v possess exactly v zeros in the interval $0 < x < 2$.

The case $\lambda < 0$ can be discussed in the same way but need not be considered here. It may be mentioned, however, that every $\lambda < 0$ is an eigenvalue, which shows that results obtained for a finite interval $0 \leqslant x \leqslant a$ cannot be transferred blindly to an infinite interval, for under the boundary conditions $y'(0) = 0$, $y(a) = 0$, (A.1) has only a countably infinite number of eigenvalues $\lambda < 0$, Kamke, 1961.

Information on the somewhat more general boundary value problem

$$z'' = [\mu(x-1)^2 - \lambda]z, \qquad z'(0) = 0, \qquad z(\infty) = 0 \tag{A.2}$$

can be obtained if for equal initial values $y(0) = z(0) \neq 0, y'(0) = z'(0) = 0$, this differential equation is compared with the differential equation (A.1), which may be written in the form

$$y'' = [\lambda(x-1)^2 - \lambda]y$$

for this purpose. Here only the least eigenvalue is of interest. If $\mu = \lambda_0$, then for $\lambda = \lambda_0$ both differential equations are identical. So λ_0 is an eigenvalue of (A.2) with $\mu = \lambda$. If μ assumes values different from λ_0, then the eigenvalue of (A.2) just found varies continuously with μ. It may be designated by $\lambda_0(\mu)$. $\lambda_0(\mu)$ thus exists in a neighbourhood of $\mu = \lambda_0$. It remains to be shown that $\lambda_0(\mu)$ is the least eigenvalue of (A.2). This may be done in the following way: If λ increases from zero, it is seen that for $\lambda < \mu$, $y'' < z''$ everywhere, so that $z(x)$ is everywhere "above" $y(x)$. Thus for $\mu \geqslant \lambda_0$ an eigenvalue of (A.2) less than λ_0 cannot exist. It follows that $\lambda_0(\mu)$ is the least eigenvalue of (A.2) and $\lambda_0(\mu) > \lambda_0$ for $\mu > \lambda_0$. For $\lambda > \mu$ we have $y'' > z''$, $z(x)$ remains below y, and $\lambda_0(\mu) < \lambda_0$ for $\lambda_0 > \mu$.

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